Transition Rates and Fermi's Golden Rule–a Cookbook

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*** MKS (SI) units ***

$$a_0 \approx \frac{\hbar}{m_e e^2} = \frac{\hbar}{m_e c\,\alpha}$$

$$\alpha = e^2/\hbar c$$

1 Brief derivation

Consider a generic process by which a quantum system (e.g. an atom) absorbs and/or emits a particle. Examples are scattering (such as the photoelectric, Compton, and Brillouin effects), radiation absorption by atoms, spontaneous emission etc. For simplicity we assume a transition involving the emission or absorption of a single particle (photon, electron, proton, neutron etc.). The "barebones" transition rate Γ_{fi} from the initial state $|\phi_i\rangle$ to the final state $|\phi_f\rangle$, obtained as the long-time limit of the transition probability per unit time, is

$$\Gamma_{fi} = \lim_{t\to\infty} \frac{d\mathcal{P}_f}{dt} \approx \frac{2\pi}{\hbar} |\langle\phi_f|\widehat{H}_1|\phi_i\rangle|^2\, \delta(E_f - E_i \pm E), \tag{1}$$

where $E_{f(i)} \equiv E^0_{f(i)}$ are the unperturbed energies and $\pm E$ is the energy exchanged during the transition ($+E$ for emission, $-E$ for absorption). The delta function expresses energy conservation. The (somewhat vague) long-time premise means that the detection occurs *well after* the duration of the perturbation—embodied by an "evolutionary time" Δt—during which energy is not necessarily conserved, having an uncertainty $\Delta E \sim \hbar/\Delta t$. Beside the fuzzyness of the "long-time" notion, there is another, more systemic, difficulty with Eq. 1. First, the delta function signifies that the transition occurs between states of sharply defined energies: $E_f = E_i - E$ (emission) or $E_f = E_i + E$ (absorption). The finite resolution of the detectors, however, imposes a spread of the total final energy in an interval $[E, E + \delta E]$. Furthermore, the energy of the particle(s) exchanged does not specify the final state completely. We typically need to know in which direction and how fast the particles move hence their momentum range $[\boldsymbol{p}, \boldsymbol{p} + \delta\boldsymbol{p}]$. Therefore we have to update the transition rate to account for the final momentum and energy spreads. The solution is conceptually simple: integrate Eq. 1 over the momentum states i.e.

$$\Gamma_{fi} = \int \frac{2\pi}{\hbar} |V_{fi}|^2 \, \delta(E_f - E_i \pm E) \, d\boldsymbol{p},$$

where $V_{fi} \equiv \langle \phi_f | \widehat{H}_1 | \phi_i \rangle$. First, we need to find the number of states with momentum in $[\boldsymbol{p}, \boldsymbol{p} + d\boldsymbol{p}]$. As usual, we discretize the space via a cubic lattice of side L. At large distances from the perturbation, the wavefunction of the emitted or absorbed particle—of momentum $\boldsymbol{p}$ and energy E—can be approximated by a normalized plane wave,

$$\xi(\boldsymbol{r}, t) = \frac{e^{i\,(\boldsymbol{p}\cdot\boldsymbol{r} - Et)/\hbar}}{\sqrt{V}},$$

where $V = L^3$ is the volume of the "box" in which the process occurs. Applying periodic boundary conditions in the cube i.e.

$$\xi(x, y, z) = \xi(x + L, y, z) = \xi(x, y + L, z) = \xi(x, y, z + L)$$

leads to

$$e^{i\,p_x L/\hbar} = e^{i\,p_y L/\hbar} = e^{i\,p_z L/\hbar} = 1.$$

Thus the momentum components of the emitted/absorbed particle satisfy

$$p_x = N_x \frac{2\pi\hbar}{L} \qquad p_y = N_y \frac{2\pi\hbar}{L} \qquad p_z = N_z \frac{2\pi\hbar}{L},$$

where $N_i \in \mathbb{Z}$. Reverting to the continuous-space representation, the momentum interval becomes

$$\Delta\boldsymbol{p} = \Delta\boldsymbol{N} \frac{(2\pi\hbar)^3}{V} \to d\boldsymbol{p} \equiv d^3p = d^3N \frac{(2\pi\hbar)^3}{V}$$

where $\boldsymbol{N} = (N_x, N_y, N_z)$. In spherical coordinates, the momentum-space volume element is

$$d^3p = p^2 dp \, d\Omega_p \tag{2}$$

where $d\Omega_p = \sin\theta_p d\theta_p d\phi_p$ is the infinitesimal p-space solid angle, indicating the direction of detection (incidence) of the emitted (absorbed) particle. The infinitesimal transition rate, accompanied by the emission/absorption of a particle with momentum $\in [\boldsymbol{p}, \boldsymbol{p} + d\boldsymbol{p}]$ and energy $\in [E, E + dE]$ is then

$$\begin{aligned} d\Gamma_{fi} &= \frac{2\pi}{\hbar} \frac{V d^3p}{(2\pi\hbar)^3} |V_{fi}|^2 \, \delta(E_f - E_i \pm E) \\ &= \frac{2\pi}{\hbar} \frac{V}{(2\pi\hbar)^3} p^2 \left(\frac{dp}{dE}\right) dE \, d\Omega_p \, |V_{fi}|^2 \, \delta(E_f - E_i \pm E) \\ &= \frac{2\pi}{\hbar} \rho(E) \, d\Omega_p \, dE \, |V_{fi}|^2 \, \delta(E_f - E_i \pm E). \end{aligned} \tag{3}$$

Note: Eq. 3 is a phase-space volume element since it involves the product of spatial and momentum coordinates ($d^3r\, d^3p \to V\, d^3p$). In the last line, the density of states

$$\rho(E)d\Omega_p \equiv \frac{d^3N}{dE} = \frac{V}{(2\pi\hbar)^3}\, p^2 \left(\frac{dp}{dE}\right) d\Omega_p$$

was introduced, denoting the number of states per unit energy. The updated transition rate is obtained finally by integrating Eq. 3 over the desired energy range and solid angle. For example, in the case of emission, one can choose between isotropic detection (i.e. particles emitted at all angles) and directed detection (i.e. over the solid angle subtended by a particular detector).

$$\Gamma_{fi} = \frac{2\pi}{\hbar} \int_{\text{det}} d\Omega_p \int dE\, \rho(E)\, |V_{fi}|^2\, \delta(E_f - E_i \pm E) \tag{4}$$

$$= \frac{2\pi}{\hbar} \int_{\text{det}} d\Omega_p \left[\rho(E)\, |V_{fi}|^2\right]_{E=\mp(E_f-E_i)}. \tag{5}$$

The factor in the brackets, obtained from the property of the delta function $\int f(x)\delta(x - x_0)dx = f(x_0)$, is evaluated at the exchange energy E specified by energy conservation. (Top signs: emission, bottom signs: absorption.)

2 Application examples

2.1 Emission/absorption of a photon

If the particle emitted or absorbed is a photon, it is useful to use the second expression of d^3p in Eq. 2, as a function of the relativistic photon energy $E = pc = \hbar kc = \hbar\omega$:

$$d^3p = p^2 \left(\frac{dp}{dE}\right) dE\, d\Omega_p = \frac{p^2}{c} dE\, d\Omega_p = \frac{E^2}{c^3} dE\, d\Omega_p = \frac{(\hbar\omega)^2}{c^3} dE\, d\Omega_p$$

2.2 Emission/absorption of a free particle

A free particle has kinetic energy only

$$E = \frac{p^2}{2m}$$

hence

$$d^3p = p^2 \frac{m}{p}\, dE\, d\Omega_p = mp\, dE\, d\Omega_p = m(2mE)^{1/2}\, dE\, d\Omega_p$$

2.3 Emission/absorption of N free particles

For generality, here we consider the possible recoil of the quantum system that absorbs or emits particles. If N particles are emitted (e.g. decays of large nuclei, multi-photon de-excitations etc.)

or absorbed (e.g. large atoms hit by photon or particle beams), the phase-space integration of Eq. 3 must now be done over *all* momenta while accounting for the conservation laws, yielding

$$\begin{aligned}\Gamma_{fi} &= \frac{2\pi}{\hbar} \underbrace{\int\int ... \int}_{N} \prod_{k=1}^{N} \rho(E_k)\, |V_{fi}|^2\, d\Omega_k dE_k \ \delta\left(E_f - E_i \pm \sum_{k'=1}^{N} E'_k\right) \delta\left(\boldsymbol{p}_f - \boldsymbol{p}_i - \sum_{k'=1}^{N} \boldsymbol{p}'_k\right) \\ &= \frac{2\pi}{\hbar} \underbrace{\int\int ... \int}_{\text{indep. momenta}} \prod_{k=1}^{N} \rho(E_k)\, |V_{fi}|^2\, d\Omega_k dE_k \ \delta\left(E_f - E_i \pm \sum_{k'=1}^{N} E'_k\right). \end{aligned} \tag{6}$$

The first version of Eq. 6 shows the energy and momentum conservation explicitly while the second version points out that the number of integrals is actually equal to the *independent* momenta. For example, if an atom or nucleus decays into three particles, only two momenta are independent; the third is automatically set by the conservation law. Nevertheless, the phase-space product $\prod_k \rho(E_k)\, |V_{fi}|^2\, d\Omega_k dE_k$ must be over *all* particles. If the quantum system undegoing the transition is much heavier than the emitted/absorbed particles, its recoil is negligible i.e. $\boldsymbol{p}_f \approx \boldsymbol{p}_i$ and the momentum conservation factor becomes irrelevant.