BLACKBODY RADIATION

A Planck's semiclassical derivation

Energy density per unit frequency emitted from hole in a cavity:

$$u(\omega,T) = \mathcal{N}(\omega)\,\langle\epsilon\rangle_\omega$$

density of modes in $\omega,\omega+d\omega$ (spectral mode density)

mean energy of oscillators in cavity walls

Premise: The EM field present in a cavity with vacuum (no particles in its volume) must be due to arrays of SHOs in the cavity wall (Planck)

$$\mathcal{N}(\omega) = \frac{8\pi\nu^2}{c^3} = \frac{8\pi\left(\frac{\omega}{2\pi}\right)^2}{c^3} = \frac{2\omega^2}{\pi c^3}$$

Photon gas (isothermal):

$$\langle\epsilon\rangle_\omega = \frac{\sum_{n=0}^{\infty}(n\hbar\omega)e^{-\beta n\hbar\omega}}{\sum_{n=0}^{\infty}e^{-\beta n\hbar\omega}} = \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}, \quad \beta = 1/k_BT$$

BE distribution

$$\Rightarrow \boxed{u(\omega,T) = \frac{2\omega^2}{\pi c^3}\,\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}}$$

spectral energy density (Planck's distribution)

B Derivation using Fermi's Golden Rule

Spontaneous emission: $\Gamma^{sp}_{21} \equiv A_{21} = \int d\Gamma^{sp}_{21}$

stimulated emission: $\Gamma^{stim}_{21} \equiv B_{21} = \int d\Gamma^{stim}_{21} \sim n_{\vec{k}s}$

stimulated ~~photoabsorption~~ absorption: $\Gamma^{stim}_{12} \equiv B_{12} = \int d\Gamma^{stim}_{12} \sim n_{\vec{k}s}$

$|E_2\rangle$ (N_2 atoms)

stim, spont, stim

$|E_1\rangle$ (N_1 atoms)

$$d\Gamma^{spont}_{21} = \frac{2\pi}{\hbar}\left|\langle E_1|\hat{H}_1|E_2\rangle\right|^2 d\Omega\,\delta(\omega_0-\omega_k) \Rightarrow A_{21}$$

$$d\Gamma^{stim}_{21} = \frac{2\pi}{\hbar}\left|\langle E_1|\hat{H}_1|E_2\rangle\right|^2 d\Omega\,\delta(\omega_k-\omega_0)\,n_{\vec{k}s} \Rightarrow B_{21}$$

$$d\Gamma^{stim}_{12} = \frac{2\pi}{\hbar}\left|\langle E_2|\hat{H}_1|E_1\rangle\right|^2 d\Omega\,\delta(\omega_k-\omega_0)\,n_{\vec{k}s} \Rightarrow B_{12}$$

Notes:

$\omega_0 \equiv (\epsilon_2-\epsilon_1)/\hbar$

Stimulated emission/absorption processes depend on pre-existing photon numbers ($n_{\vec{k}s} = \hat{a}^\dagger_{ks}\hat{a}_{ks}$)

Detailed balancing (equilibrium): N_1, N_2 no. of atoms in states $|1\rangle, |2\rangle$

$$0 = N_1B_{12} - N_2(A_{21}+B_{21}) = N_1 n_{\vec{k}s} - N_2(1+n_{\vec{k}s})$$

$$\Rightarrow \frac{N_1}{N_2} = \frac{n_{\vec{k}s}+1}{n_{\vec{k}s}} \Rightarrow \boxed{n_{\vec{k}s} = \frac{1}{\frac{N_1}{N_2}-1} = \frac{1}{\frac{e^{-\beta\epsilon_1}}{e^{-\beta\epsilon_2}}-1} = \frac{1}{e^{+\beta\hbar\omega_k}-1}}$$

$\hbar\omega_k \equiv \epsilon_2-\epsilon_1$

$N_n = e^{-\beta\epsilon_n}$

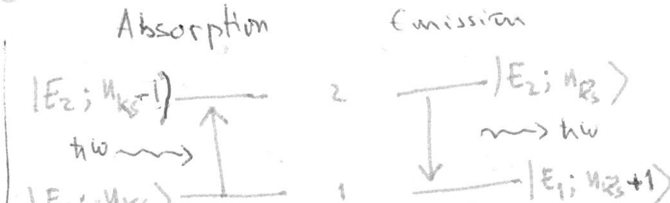
$\beta = 1/k_BT$

~~Mean~~ Total "photon gas" energy: $E_{/\vec{k}s} = n_{\vec{k}s} \hbar \omega_k = \dfrac{\hbar\omega_k}{e^{+\frac{\hbar\omega_k}{k_BT}} - 1} = \epsilon_{\vec{k}s}(\omega_k)$

$\Rightarrow$ spectral energy density:

$$\boxed{u(\omega_k, T) = \mathcal{N}(\omega_k)\,\epsilon_{\vec{k}s}(\omega_k) = \frac{2\omega_k^2}{\pi c^3}\,\frac{\hbar\omega_k}{e^{+\beta\hbar\omega_k} - 1} = \frac{2\hbar\omega_k^3}{\pi c^3\left(e^{\beta\hbar\omega_k} - 1\right)}}$$

$\boxed{C}$ Derivation similar to $\boxed{B}$, using the second quantization explicitly



$$\hat{H}_1 = \frac{-e}{m}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\left[\hat{a}_{\vec{k}s}\, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)}\, \vec{e}_{\vec{k}s}\cdot\hat{\vec{p}} - \hat{a}^{+}_{\vec{k}s}\, e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)}\cdot \vec{e}^{\,*}_{\vec{k}s}\cdot\hat{\vec{p}}\right]$$

* Stimulated absorption: $1 \to 2 \left(\Gamma_{12}^{stim} \equiv B_{12}\right)$

$|i\rangle = |\epsilon_1; n_{\vec{k}s}\rangle \equiv |\epsilon_1\rangle \otimes |n_{\vec{k}s}\rangle$

$|f\rangle = |\epsilon_2; n_{\vec{k}s}-1\rangle \equiv |\epsilon_2\rangle \otimes |n_{\vec{k}s}-1\rangle$

$\langle f|\hat{H}_1|i\rangle \sim \langle f|\hat{a}_{ks}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}_{ks}\cdot\hat{\vec{p}} - \hat{a}^{+}_{ks}\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}^{\,*}_{ks}\cdot\hat{\vec{p}}\,|i\rangle$

$\boxed{\text{Work at } t = 0}$

$= \langle\epsilon_2; n_{ks}-1|\hat{a}_{ks}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_1; n_{\vec{k}s}\rangle$

$- \langle\epsilon_2; n_{ks}-1|\hat{a}^{+}_{ks}\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}^{\,*}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_1; n_{ks}\rangle =$

$= \langle\epsilon_2|e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_1\rangle \underbrace{\langle n_{ks}-1|\hat{a}_{ks}|n_{ks}\rangle}_{\sqrt{n_{ks}}\,|n_{ks}-1\rangle} - \langle\epsilon_2|e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}^{\,*}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_1\rangle \underbrace{\langle n_{ks}-1|\hat{a}^{+}_{ks}|n_{ks}\rangle}_{=0}$

$= \langle\epsilon_2|e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_1\rangle\, \sqrt{n_{ks}} \quad \Rightarrow \quad B_{12} \sim \left|\langle E_2|e^{i(\vec{k}\cdot\vec{r})}\, \vec{e}_{ks}\cdot\hat{\vec{p}}\,|E_1\rangle\right|^2 n_{ks}$ ← (1)

* Stimulated & spontaneous emission: $2 \to 1$ $\left(\Gamma_{21} = \Gamma_{21}^{stim} + \Gamma_{21}^{spon} \equiv B_{21} + A_{21}\right)$

$|i\rangle = |\epsilon_2; n_{ks}\rangle$; $|f\rangle = |\epsilon_1; n_{ks}+1\rangle$ ← stim. emission

$\langle f|\hat{H}_1|i\rangle \sim \langle\epsilon_1; n_{ks}+1|\hat{a}_{ks}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_2; n_{ks}\rangle - \langle\epsilon_1; n_{ks}+1|\hat{a}^{+}_{ks}\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}^{\,*}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_2; n_{ks}\rangle$

$= \langle\epsilon_1|e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_2\rangle \underbrace{\langle n_{ks}+1|\hat{a}_{ks}|n_{ks}\rangle}_{=0} - \langle\epsilon_1|e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}^{\,*}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_2\rangle \underbrace{\langle n_{ks}+1|\hat{a}^{+}_{ks}|n_{ks}\rangle}_{\sqrt{n_{ks}+1}\,|n_{ks}+1\rangle}$

$= -\langle\epsilon_1|e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, \vec{e}^{\,*}_{ks}\cdot\hat{\vec{p}}\,|\epsilon_2\rangle\, \sqrt{n_{ks}+1}$

$\Gamma_{21} \equiv \Rightarrow B_{21} + A_{21} \sim \left|\langle E_1|e^{-i\vec{k}\cdot\vec{r}}\, \vec{e}^{\,*}_{ks}\cdot\hat{\vec{p}}\,|E_2\rangle\right|^2 (n_{ks}+1)$ ← (2)

total emission rate → Then, detailed balancing $N_1 B_{12} = N_2(B_{21} + A_{21}) \Rightarrow n_{ks} = \dfrac{1}{e^{\beta\hbar\omega} - 1}$ (see $\boxed{B}$ p. LAI 12)